Synchronization of the Frenet–Serret Linear System with a Chaotic Nonlinear System by Feedback of States

G. Solis-Perales,[1] H.C. Rosu,[2] and C. Hernandez-Rosales[2,3]

A synchronization procedure of the generalized type in the sense of Rulkov *et al*. [Phys. Rev. E 51, 980 (1995)] is used to impose a nonlinear Malasoma chaotic motion on the Frenet–Serret system of vectors in the differential geometry of space curves. This could have application in biological molecular motion.

KEY WORDS: Frenet–Serret equations; Malasoma chaos; Lie derivatives; synchronization.

The Frenet–Serret (FS) vector set is in much use whenever one focuses on the kinematical properties of space curves. The evolution in time of the FS triad is one of the most used descriptions of the motion of tubular structures such as stiff (hard to bend) polymers Kamien (2002). Many biological polymers including the DNA helical molecule are stiff and their movement is of fundamental interest. At the mesoscopic level many sources of noise and chaotic behaviour affect in a substantial way the motion of the biological polymers. In general, one can think that a synchronization between the motion of the polymers and the chaotic (or noisy) sources could be achieved in a natural way through some control signal. We illustrate this idea employing a generalized synchronization procedure based on the theory of nonlinear control by which we generate a chaotic dynamics of the FS evolution equations

$$\dot{\vec{T}} = \kappa \cdot \vec{N}$$
$$\dot{\vec{N}} = \tau \cdot \vec{B} - \kappa \cdot \vec{T}$$

[1] Universidad Autonoma de Yucatan, Apdo Postal 150, Cordemex 97310 Mérida, Yucatán, México.

[2] Departamento de Matemáticas Aplicadas y Sistemas Computacionales, IPICYT, Apdo. Postal 3–74 Tangamanga, 78231 San Luis Potosí, Mexico.

[3] To whom correspondence should be addressed at Departamento de Matemáticas Aplicadas y Sistemas Computacionales, IPICYT, Apdo. Postal 3–74 Tangamanga, 78231 San Luis Potosí, Mexico; e-mail: hcr@ipicyt.edu.mx.

$$\dot{\vec{B}} = -\tau \cdot \vec{N}.$$

With this goal in mind, one should first write the FS system in the form

$$\dot{x} = Ax + Eu \qquad\qquad y = Cx, \qquad\qquad (1)$$

where $x \in R^3$ is the vector having as components the tangent T, normal N and the binormal B, whereas A is the transfer matrix, E is an initial vector that determines the channel where the control signal is applied, and C determines the measured signal of the FS system.

In this way, the main objective from the synchronization standpoint is to force the states of the slave system, which is the FS system to follow the trajectories of the master system that, in general, presents a chaotic behaviour. This is achieved by applying a signal given by $U = \Phi(x)$. We first rewrite the FS system in the form
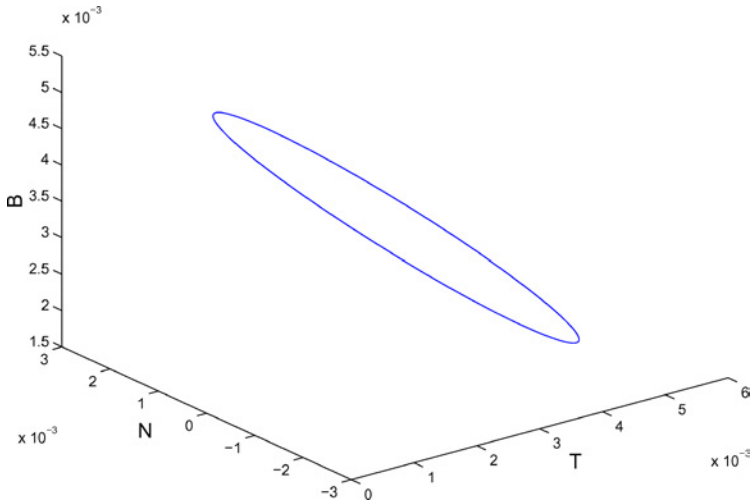
$$\begin{pmatrix} \dot{\vec{T}} \\ \dot{\vec{N}} \\ \dot{\vec{B}} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix} + \begin{pmatrix} \vec{0} \\ \vec{0} \\ \vec{1} \end{pmatrix} \Phi$$

$$y = (c_1 \ c_2 \ c_3) \begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix} \qquad\qquad (2)$$

where $c_1 = 1$, $c_2 = 0$, $c_3 = 1$. If we choose now $B = (0\ 0\ 0)^T$ as the input vector, one gets the open loop FR dynamics. This type of dynamics is shown in Fig. 1 for $\kappa = 1$ and $\tau = 0.9$ and initial conditions given by $T_0 = 0.0024$, $N_0 = 0.0026$, and $B_0 = 0.0039$, where the states of the system are given by the derivatives of the tangent, normal, and binormal unit vectors, respectively, whereas κ and τ are the curvature and torsion scalar invariants.

On the other hand, a chaotic oscilator is a dynamic system whose evolution is difficult to predict. In general, its main feature is the sensibility to the initial conditions and the variations of the parameters. Thus, its long-term behaviour is hard to estimate. In the following we use one of the simplest chaotic oscilator system introduced by Malasoma (2002)

$$\begin{aligned} \dot{X}_1 &= X_2 \\ \dot{X}_2 &= X_3 \\ \dot{X}_3 &= -\alpha X_3 - X_1 + X_1 X_2, \end{aligned} \qquad\qquad (3)$$

Fig. 1. Three-dimensional dynamics of the Frenet–Serret system with $\kappa = 1$ and $\tau = 0.9$.

where α is the bifurcation parameter. In matrix form, we get

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & X_1 & -\alpha \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

$$y = (1\ 0\ 0)x. \tag{4}$$

This system exhibits chaotic behaviour for $2.0168 < \alpha < 2.0577$. The chaotic evolution is shown in Fig. 2 for $\alpha = 2.025$ and initial conditions $X_1 = 0.0022$, $X_2 = 0.0024$, $X_3 = 0.0039$. Having the two systems in the matriceal form, we choose the Malasoma one as the master system and the FS system as the slave. The Malasoma dynamics is imposed to the FS motion through the signal $U = \Phi(x)$. In other words, a nonlinear dynamics is forced upon the FS system leading to its chaotic behaviour.

To get the chaotic FS system one should achieve the synchronization between the master and the slave systems. For this, one defines a third system, which refers to the synchronization error given by the difference in the dynamics of the two systems, i.e.,

$$\dot{e}_1 = \kappa e_2 + \lambda_1(X_M)$$

$$\dot{e}_2 = \tau e_3 - \kappa e_1 + \lambda_2(X_M) \tag{5}$$

$$\dot{e}_3 = -\tau e_2 + \lambda_3(X_M) - U,$$

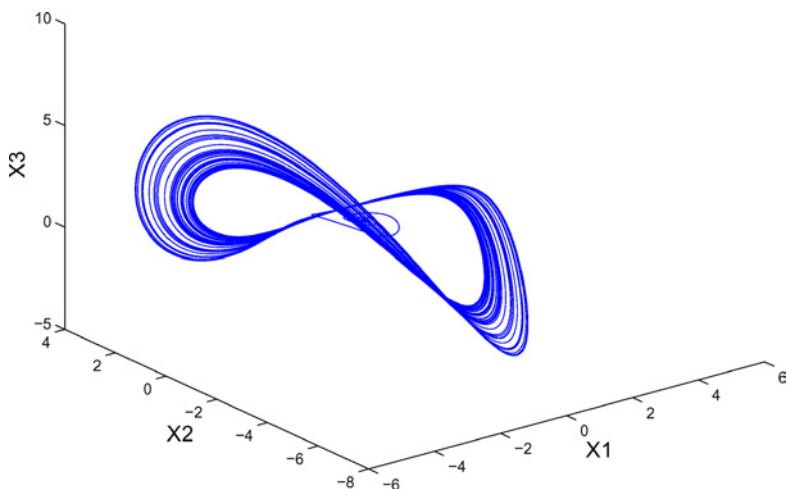**Fig. 2.** Three-dimensional dynamics of the Malasoma system with $\alpha = 2.025$.

where

$$\lambda_1 = X_{2M}(1 - \kappa),$$

$$\lambda_2 = X_3(1 - \tau) + \kappa X_{1M} \tag{6}$$

$$\lambda_3 = -\alpha X_{3M} + X_{1M}(X_{2M} - 1) + \tau X_{2M}.$$

The function $U = \Phi(e)$ gives the control action that leads to the synchronization of the two systems.

Once (5) defined, one simply chooses $y = C = (1\ 0\ 0)e$ (or $y = e_1$) as the output of the error system. In the synchronization approach, one writes $y = h(e) = e_1 = X_{1M} - X_{1S}$, and consequently the error system (5) can be written in the general form

$$\dot{X} = f(e) + g(e)U. \tag{7}$$

The error system (7) should be stabilized at the origin or in an arbitrary small neighborhood of it. More details on the synchronization conditions are provided in the papers (Femat and Alvarez-Ramirez, 1997; Solis-Perales *et al.*, 2003) and can be employed to obtain the control function (Isidori, 1989)

$$U = \frac{-1}{\gamma}(\beta + \delta), \tag{8}$$

where γ and δ are real-valued functions obtained by means of Lie derivatives of $h(e)$ as follows

$$\gamma = L_g L_f^{\rho-1} h(e), \qquad \beta = L_f^{\rho} h(e) \tag{9}$$

where ρ is a positive integer that determines the relative degree of the system (see Isidori, 1989). On the other hand, the desired dynamics, i.e., directed towards the origin, is dictated by

$$\delta = K_{P1}e_1 + K_{P2}e_2 + K_{P3}e_3 \tag{10}$$

Thus, performing the Lie derivatives and regrouping the terms, we obtain U of the form

$$U = \frac{-1}{\kappa\tau}[(-\kappa^2\dot{e}_1 + \kappa\tau\dot{e}_3) + K_{P1}e_1 + K_{P2}e_2 + K_{P3}e_3]. \tag{11}$$

Using the change of variables $e_i = X_{iM} - X_{iFS}$, where the latter vector is the column vector formed by the triad T, N, and B ($i = 1, 2, 3$), the control can be written as a function of the states of the two systems

$$U = \Phi(x) = \frac{-1}{\kappa\tau}[-\kappa^2(X_2 - \kappa N) - \kappa\tau[(-\alpha X_3 - X_1 + X_1 X_2) + (\tau N)] + \delta], \tag{12}$$

where $\delta = K_{P1}(X_1 - T) + K_{P2}(X_2 - N) + K_{P3}(X_3 - B)$. Notice that $\gamma = L_g L_f^2 h(x) = \kappa\tau$ is a nonzero constant. Therefore, the control signal is defined for any T,N,B, X_1,X_2, and X_3. In addition, one should choose the values of the
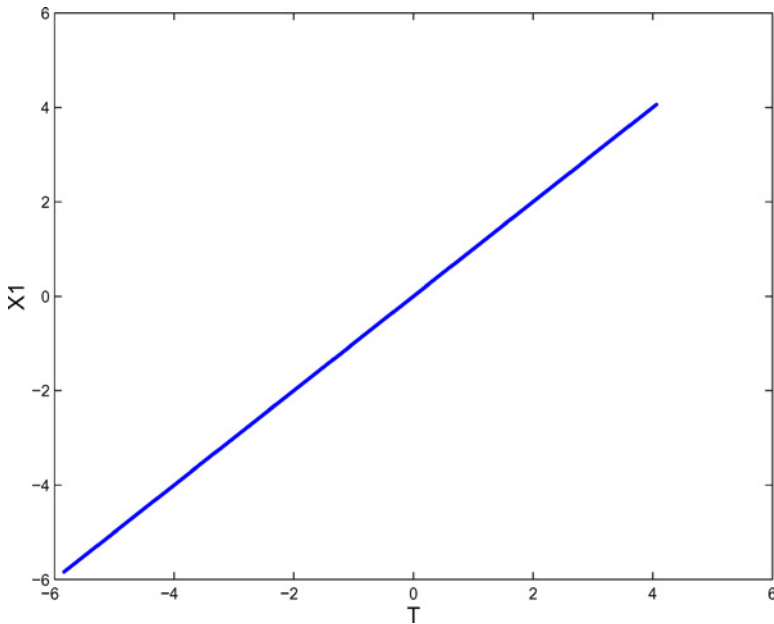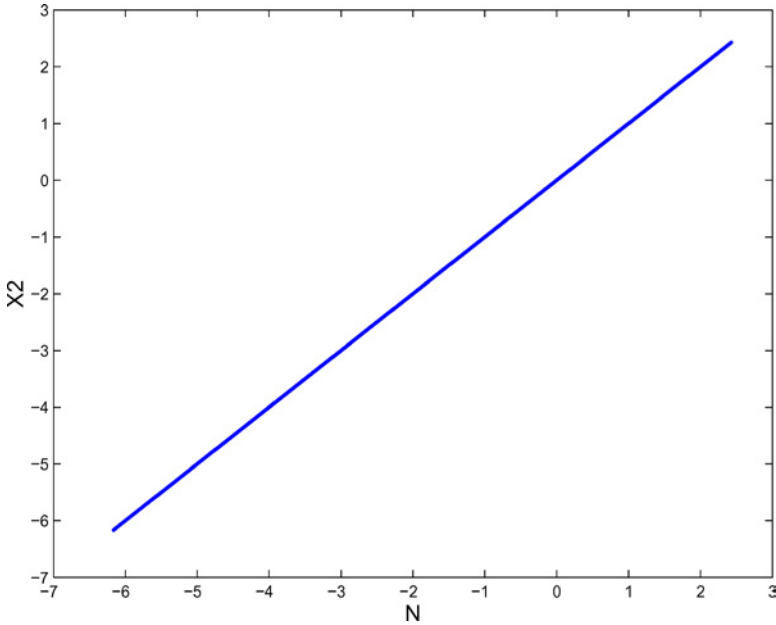

Fig. 3. Phase locking of the T states of the FS system to the X_1 states of the Malasoma system.

Fig. 4. Phase locking of the N states of the FS system to the X_2 states of the Malasoma system.

constants K'_{Pi} in such a way that the differences in δ go to zero. Applying the dynamics generated by (12) leads to the synchronization matrix

$$X_{\text{FS}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \kappa & 0 & 1 \end{pmatrix} \begin{pmatrix} X_{1\text{M}} \\ X_{2\text{M}} \\ X_{3\text{M}} \end{pmatrix}. \tag{13}$$

From this synchronization matrix one can see that the first two states of both systems are synchronized. However, the state $B = X_{3\text{M}} + \kappa X_{1\text{M}}$ is expanded by the term $\kappa X_{1\text{M}}$, i.e., the state B is the sum of two of the Malasoma oscillator; since the latter is chaotic, one concludes that the state B is also chaotic.

We display the phase locking between the corresponding phases of the two oscillators in Figs. 3 and 4 where the phase locking of the states T and X_1 and N and X_2, respectively, shows that the two pairs of states are synchronized. In Fig. 5, we see that the B and X_3 states are not synchronized. Thus, following the terminology of (Rulkov *et al.*, 1995), we are in the situation of a generalized synchronization. In Fig. 6 the two already synchronized systems are shown in the three-dimensional space. One can notice that the FS system is 'above' the Malasoma oscillator, and that the two systems are in a chaotic phase. Finally, in
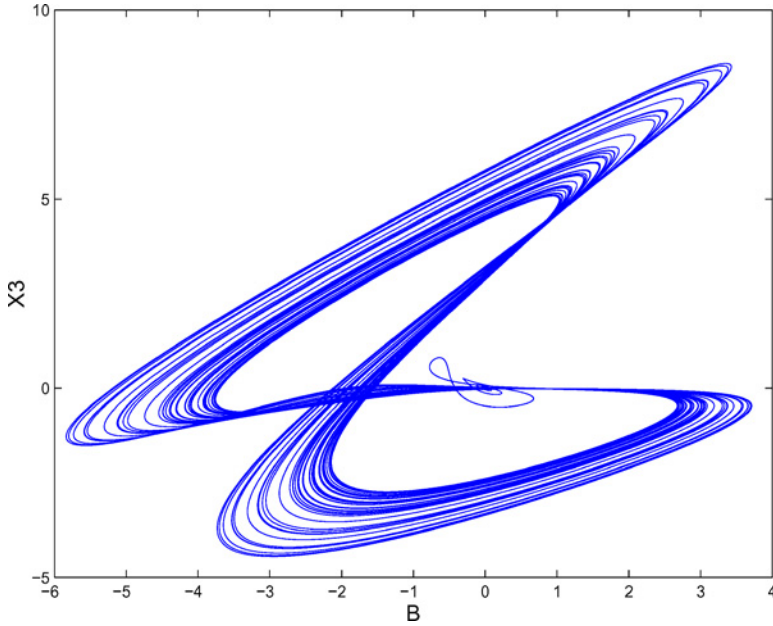
Fig. 5. Phase behaviour of the states B and X_3 of the FS system and Malasoma system, respectively.
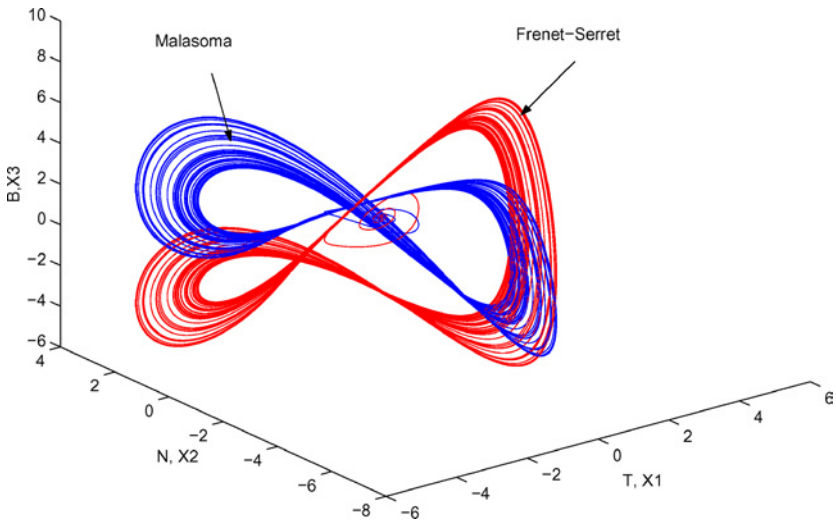
Fig. 6. Plot of the two chaotic attractors once they are synchronized in the generalized form.
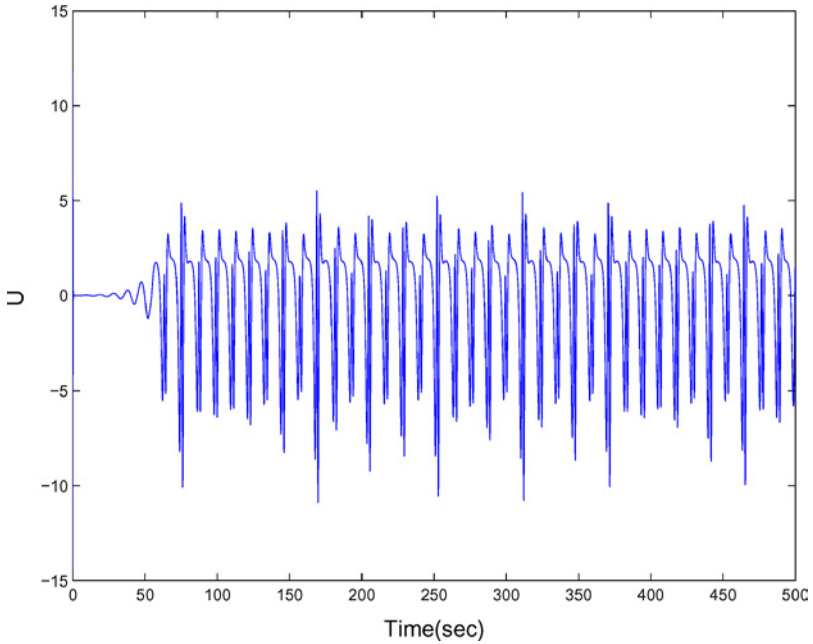
Fig. 7. The control signal U imposed to the Frenet–Serret system in order to display chaos.

Fig. 7, the control signal used to achieve the generalized synchronization of this paper is displayed.

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